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Wave Propagation in Strongly Coupled Classical Plasmas in an External Magnetic Field

R. O. Genga[1]

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When a small perturbation is applied to the plasma dispersion, a small shift of frequency due to correlation occurs. This is justified even for strong coupling, since the effect is proportional to k^2 (k is the wave vector) and it is sufficient to consider the $k \to 0$ limit. Then by solving the dispersion relations for $\delta\omega$, the shift of frequency due to correlation, at different angles of propagation, we obtain all information needed. The plasma modes in which we are primarily interested are the "whistler" and the "extraordinary" modes. In this work the STLS (Singwi, Tosi, Land, and Sjolander) approximation scheme is used. It is seen that the correlational effects enter only through terms of order k^6 for the whistler mode and terms of order k^2 for the nonresonant situation of the extraordinary mode.

1. INTRODUCTION

In weakly correlated plasmas the potential energy of the particles is much less than their kinetic energy (Genga, 1986). Thus, in general, the contribution of the potential energy of the particles is neglected when their response is considered to excitations due to external or self-consistent fields. In cases, however, where the potential energy of the particles is equal to or greater than their kinetic energy (strongly coupled plasmas) a special approach is called for. In this work, we consider a nonrelativistic, homogeneous, one-component classical plasma in equilibrium in the presence of a uniform external magnetic field; the corresponding quantum plasma case can be published soon.

Three major approximation schemes can be used to obtain the dispersion properties of strongly coupled plasmas: the STLS (Singwi, Tosi, Land, and Sjolander), the TI (Totsuji and Ichimaru), and the GKS (Golden, Kalman, and Silevitch) schemes, respectively. Correlation effects on transverse modes only occur through longitudinal-transverse coupling, since all

[1]Department of Physics, University of Nairobi, Nairobi, Kenya.

these schemes consider longitudinal internal field only. It is known (Carini *et al.*, 1980) that for strong coupling, longitudinal plasma modes exhibit "negative dispersion." The main interest in studying transverse modes lies in determining whether a similar change takes place and what the critical coupling value is. The problem is related to the Malmberg-O'Neil experiment, where a strongly magnetized, strongly coupled plasma is generated.

In this work we use the STLS schemes (Kalman, 1978) for the following reasons:

1. The problem for a magnetized plasma is rather complex because of the anisotropic character of the system, which demands that one use all six elements of the dielectric tensor; therefore a simple approximation is called for.
2. One knows that for an unmagnetized plasma the STLS gives qualitatively reasonable results for the plasma frequency shift even though it does not for the plasma damping. Since in our work the primary objective is to determine the frequency shift in various modes, the STLS seems to be adequate.
3. The STLS, or any mean field theory, is reasonably good for the low-frequency modes of interest to us.

In Section 2 we considered the perturbation method used in this work. Sections 3 and 4 comprise the main body of our work. We determine the frequency shift due to correlations of the whistler and the extraordinary modes for arbitrary direction of propagation as a function of coupling parameter γ. In the latter case, one has to distinguish between the "nonresonant" and "resonant" situations, depending upon whether the cutoff frequency ω_1 is different from or coincides with the cyclotron frequency.

In Section 3 we consider the case of a cold plasma; temperature corrections are obtained in Section 4. We also study the effects of damping on the modes under consideration.

2. THE PERTURBATION METHOD

A shift of frequency due to correlation, $\delta_u\omega$, occurs when a small perturbation is applied to the dispersion relations. The correlation is very weak for weakly coupled plasmas, but can be strong for strongly coupled plasmas. The shift $\delta_u\omega$ is of order k^2, and thus is small as $k \to 0$, which is equal to the order of the frequency shifts $\delta_n\omega$ and $\delta_t\omega$ caused by refractive and thermal effects, respectively, even for $\gamma \gg 1$. That is, after perturbation we find that

$$\omega = \omega^0 + \delta\omega \tag{1}$$

where ω^0 is the unperturbed frequency, defined as

$$\omega^2 = \begin{cases} (\Omega k^2 c^2/\omega_p^2)\cos\theta & \text{(Whistler mode)} \\ \omega_1 & \text{(low-frequency extraordinary mode)} \end{cases} \quad (2)$$

with ω_1, ω_p, and Ω defined as

$$\begin{aligned} \omega_1 &= \tfrac{1}{2}\Omega^2[-1+(1+4\omega_p^2/\Omega^2)^{1/2}] \\ \omega_p &= (4\pi e^2 n/m)^{1/2} \qquad \text{(electron plasma frequency)} \\ \Omega &= \frac{eB}{m} \qquad \text{(electron cyclotron frequency)} \end{aligned} \quad (3)$$

and $\delta\omega$ is the total shift in frequency caused by refractive, thermal, and correlational effects. That is, in strongly coupled plasmas,

$$\delta_\omega = \begin{cases} \delta_n\omega + \delta_u\omega, & T=0 \\ \delta_n\omega + \delta_t\omega + \delta_u\omega, & T \text{ finite but small} \end{cases} \quad (4)$$

while for the case of weakly coupled plasmas we set $\delta_u\omega = 0$ in equation (4).

The dispersion relation for the plasma modes is given by

$$\Delta = 1\varepsilon - n^2\mathbf{T}1 = 0 \quad (5)$$

When a small perturbation is applied in the vicinity of a mode that exists without the perturbation, we obtain

$$\Delta_1(k\omega^0, \theta) + \delta\omega\,\Delta_0^1(\omega^0, \theta) = 0 \quad (6)$$

where

$$\Delta_0^1(\omega^0, \theta) = \left.\frac{\partial\Delta_0(\omega, \theta)}{\partial\omega}\right|_{\omega=\omega^0} \quad (7)$$

and Δ is a function of strongly coupled polarizabilities (Genga, in press). In the case of the low-frequency extraordinary mode we can have a situation where $\omega_1 = n\Omega$ (n is an integer), i.e., the resonant situation. In this situation we apply Taylor's (power series) expansion to $Z((ka\cos\theta)/(\omega - n\Omega))$, where $Z(z_n)$ is the plasma function (Fried and Cote, 1971), for $n = 0$, and apply an asymptotic expansion to the case where $n \neq 0$ when evaluating the components of weakly coupled dielectric tensor (Furutani and Kalman, 1965); physically, n stands for the nth harmonic of the resonant frequency in classical plasmas.

3. COLD PLASMAS ($T = 0$ K)

It is known that the appearance of damping and harmonics is a consequence of finite plasma temperature for the case of a classical plasma.

Therefore, for a cold plasma we have to consider the whistler mode and the nonresonant situation of the extraordinary mode only for the cases with no damping waves propagating parallel, perpendicular, or at an oblique angle to the magnetic field, respectively.

From equation (A15) of the Appendix we see that the longitudinal component of the polarizability tensor has a strongly correlated contribution for parallel propagation. Since there is no transverse-longitudinal coupling in this case, this implies that there is no coupling effect on the plasma dispersion for either the whistler mode or the nonresonant case of the extraordinary mode. The resonant situation for the extraordinary mode still does not exist (Genga, 1986). It is also known that the whistler mode does not exist when the direction of propagation is perpendicular to the magnetic field. These results also hold for the case of warm plasmas in the presence of an external magnetic field.

3.1. Whistler Mode

For this mode we only consider the propagation of waves at an oblique angle to the external magnetic field. When a small perturbation is applied to the dispersion relation, the frequency shift of the mode is given by (Figure 1)

$$\delta(\omega^2) = -\frac{\Omega^2 k^6 c^6}{\omega_p^8}\Bigg[(\omega_p^2+\Omega^2)(1+\cos^2\theta)+\omega_p^2\sin^2\theta+\frac{2\gamma\omega_p^2}{3\Omega^2\kappa^2c^2}(\omega_p^2+\Omega^2) \times[(\omega_p^2+\Omega^2)\cos^2\theta+\omega_p^2\sin^2\theta]\sin^2\theta\cos^2\theta\Bigg]\cos^2\theta \tag{8}$$

where $\gamma=\kappa^3/4\pi^2 n$ is the ratio of potential energy to kinetic energy, $\kappa^2=14\pi e^2 n\beta$, $n=N/v$ is the particle number density, and $\beta=1/k_B T$ (k_B is the Boltzmann constant). Equation (9) shows that $\delta_u\omega$ is angle-dependent. It also shows that γ should be very large, i.e., of the order $\kappa^2c^2\omega_p^{-2}$, in order for the strong coupling effect to be realized.

3.2. Nonresonant Case of the Extraordinary Mode

In this case we consider the correlation effects on modes propagating perpendicular and at an oblique angle θ to the external magnetic field. For perpendicular propagation the frequency shift is of the form

$$\delta\omega=\frac{k^2c^2}{2(2\omega_1+\Omega)}\left[\frac{(\omega_1+\Omega)^2}{\omega_p^2}+\frac{\gamma\omega_p^2}{3\kappa^2c^2}\right] \tag{9}$$

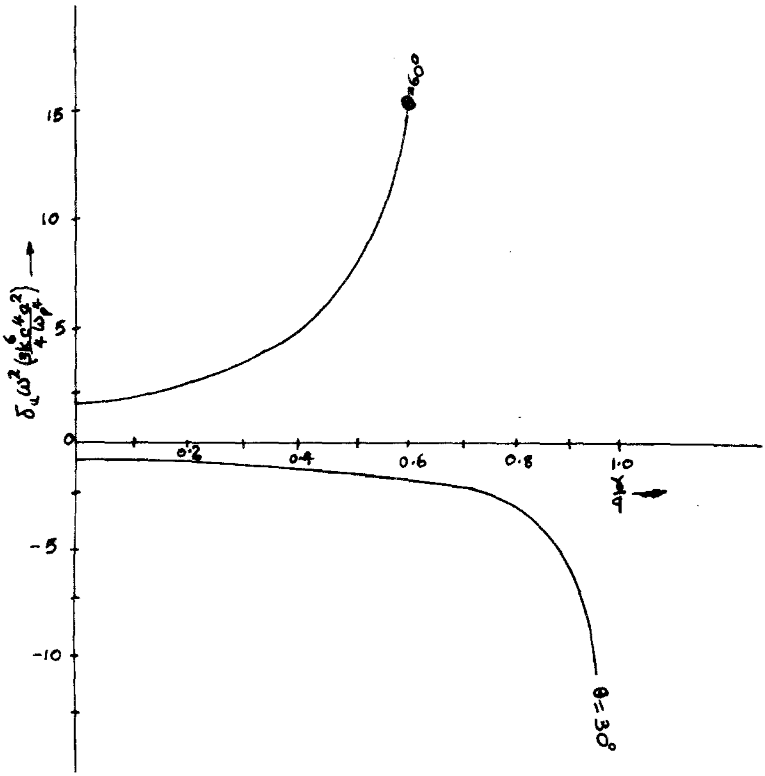

Fig. 1. Whistler mode. Strong coupling term $\delta_u\omega^2$ (in units of $3k^6c^4a^4/4\omega_p^4$) versus $\gamma/9$ for $\theta=30°$ and $60°$.

Equation (9) shows that the frequency shift of order k^2 does not have a resonance. It further shows that the frequency shift due to correlation is positive and the value of γ should be very large, i.e., of order $\kappa^2c^2\omega_p^{-2}$, in order for the strong coupling effect to be detectable.

When the direction of propagation is at an oblique angle to the external magnetic field we find that

$$\delta\omega=\frac{k^2c^2}{2\omega_1+\Omega}\left[\frac{\omega_p+\Omega}{2\omega_p^2}(1+\cos^2\theta)+\frac{\gamma\omega_p^4(\omega_1-\Omega)^2\sin^2\theta}{3\omega_1^2(\omega_1^2-\Omega^2)^2\kappa^2c^2}\right] \quad (10)$$

This shows that $\delta_u\omega$ is angle dependent and positive for all values of θ. In this situation the strong coupling effect noticable only for large value of γ as in the above cases.

4. WARM CLASSICAL PLASMAS: PLASMAS WITHOUT DAMPING

4.1. Whistler Mode

In this case the frequency shift for an arbitrary angle of propagation is given by

$$\delta(\omega^2) = -\frac{\Omega^4\kappa^6c^6}{\omega_p^4}\left\{\left(1+\frac{\omega_p^2}{\Omega^2}(1-4\tan^2\theta)\right)\cos^2\theta+\left(1+\frac{\omega_p^2}{\Omega^2}\right)\right.$$
$$+\frac{\omega_p^4a^2}{\Omega^2c^2}\left[(2\cos^2\theta-3\sin^2\theta)-\tfrac{4}{3}\gamma(\cos^2\theta-2\sin^2\theta)\right.$$
$$\left.\left.+\frac{\gamma^2}{9}(2\cos^2\theta-5\sin^2\theta)\right]\left(1-\frac{2\gamma}{3}+\frac{\gamma^2}{9}\right)^{-1}\right\}\cos^2\theta \tag{11}$$

This equation leads to a conclusion that $\delta_\mu(\omega^2)$ is infinite at $\gamma=3$. For $\gamma\neq 3$, $\delta_\mu(\omega^2)$ is either negative or positive depending on the angle of propagation as shown in Figure 1. It can further be seen from equation (11) that the correlational effect can be realized if γ is very large.

4.2. Nonresonant Case of Extraordinary Mode

For propagation perpendicular to the external magnetic field, we find that the frequency shift is of the form

$$\delta\omega = \frac{(\omega_1+\Omega)^2k^2c^2}{2(2\omega_1+\Omega)\omega_p^2}\left\{1+\frac{\omega_p^2a^2}{(\omega_1+\Omega)c^2}\left[\frac{4}{(\omega_1+2\Omega)}+\frac{\gamma}{9(\omega_1+\Omega)}\right]\right\} \tag{12}$$

Equation (12) shows that $\delta\omega$ does not have a resonance for a frequency shift of order k^2. However, the frequency shift of order k^4, i.e., the second-order frequency shift, becomes infinitely large as we approach the neighborhood of Ω and 2Ω (Genga, 1986), so that we cannot ignore its existence.

Turning to the situation when propagation is at an oblique angle θ to an external magnetic field, we find that

$$\delta\omega = \frac{\omega_1+\Omega)k^2c^2}{2(2\omega_1+\Omega)\omega_p^2}\left[(\omega_1+\Omega)(1+\cos^2\theta)+\omega_p^2\frac{a^2}{c^2}\frac{4}{\omega_1+2\Omega}\right.$$
$$\left.+\frac{\gamma}{9(\omega_1+\Omega)}\sin^2\theta+2\frac{\omega_1}{(\omega_1+\Omega)^2}\cos^2\theta\right] \tag{13}$$

In this case we see again that $\delta\omega$ does not have a resonance for a frequency shift of order less than k^4 as in the perpendicular propagation situation. Figure 2 shows that $\delta_\mu\omega$ is positive and angle-dependent.

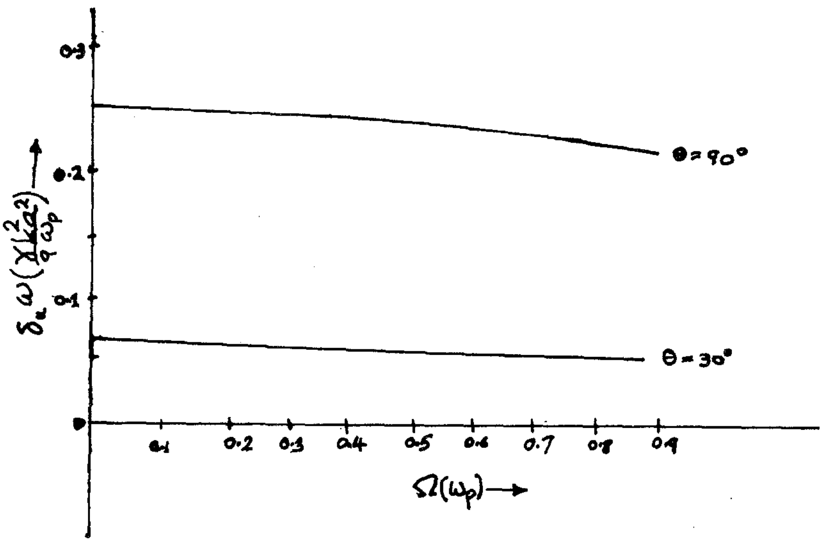

Fig. 2. Extraordinary mode, strong coupling nonresonant term $\delta_u \omega$ (in units of $\gamma k^2 a^2/9\omega_p$) versus Ω (in units of ω_p) for $\theta = 30°$ and $90°$.

4.3. Resonant Case of the Extraordinary Mode

In this case, only $n = 1$ is considered. By combining equations (6) and (A15) we obtain the elements of a strongly coupled dielectric tensor when we assume that

$$U_k(\alpha_{11}^0 \sin^2 \theta + 2\alpha_{13}^0 \sin \theta \cos \theta + \alpha_{33}^0 \cos \theta) \ll 1$$

where α_{11}^0, α_{13}^0, and α_{33}^0 are weakly coupled polarizabilities. The correlational terms in this situation are found to be of order k^2 less than the correlationless ones. Since in this case we determine the frequency shift of the first harmonics, which is of order k^2, the correlational effects are negligible because they are of order k^4. However, if the frequency shift of the higher orders is determined, the correlational effects cannot be neglected.

5. PLASMAS WITH DAMPING

From equation (A15) we find that both coupling and damping effects on waves propagating along the magnetic field are absent. It is also known that both the whistler mode and the resonance situation of the extraordinary mode do not exist for waves propagating across an external magnetic field; the nonresonant extraordinary mode exists, but it is unaffected by damping. For propagation at an oblique angle to an external magnetic field we find that the nonresonant and resonant cases of the extraordinary mode are not affected by both damping and correlations. However, for a frequency shift of order greater than k^2 of the resonant situation of the extraordinary mode the correlational terms contribute. The nonresonant situation is not affected by correlations for all values of k. The whistler mode in this case is damped, but is not affected by correlations for any value of k. In this situation where the modes are not affected by correlations we recover the results for the weakly coupled plasma case.

6. CONCLUSION

We found that for both cold plasmas and warm plasmas without damping, the coupling term is of order $\gamma a^2 c^{-2}$. This implies that in order for the coupling effect to be effective, γ has to be very large, i.e., about of order $a^{-2}c^2$. We also found that for $\gamma = 3$, $\delta(\omega^2)$ of the whistler mode for warm plasmas blows up, in contrast to $\delta(\omega^2)$ of the whistler mode for cold plasmas. This originates from the coupling with the longitudinal mode and is the result of the infinite compressibility at $\gamma = 3$. When damping is taken into account, we found that the damping of the whistler mode is not affected by coupling, even for oblique propagation. The k-independent cutoff frequencies (ω_0, ω_1) are necessarily unaffected by the coupling, since it is at least of order k^2. Somewhat less obvious is the result that even the lowest order whistler dispersion relation $(\omega^2 \cong k^4)$ remains unaffected, and correlational effects enter only through terms of order k^6. However, for the nonresonant situation of the extraordinary mode the correlational effects enter through terms of order k^2.

APPENDIX. COUPLING PROJECTION OPERATOR

When a system is under the influence of an external perturbation, such as an electric field, its behavior is characterized by response functions. Once an external field is set up, the system's response also generates additional fields, called "plasma" or "polarization" fields. These add to the external field, so that the particles respond to the total field, and physical response relates to the later.

Therefore, if $\hat{\bar{E}}$, $\hat{n}$ refer to the external field and density, $\check{\bar{E}}$, $\check{n}$ are notations for plasma field and density, and $\bar{E}$ and n stand for total field and density, then

$$E(\bar{k}\omega) = \hat{\bar{E}}(k\omega) + \check{\bar{E}}(\bar{k}\omega)$$
$$n(\bar{k}\omega) = \hat{n}(\bar{k}\omega) + \check{n}(\bar{k}\omega) \tag{A1}$$

$\bar{E}$, $\hat{\bar{E}}$, and $\check{\bar{E}}$ are interrelated as follows:

$$\bar{E}(\bar{k}\omega) = \varepsilon^{-1}(k\omega)\hat{\bar{E}}(k\omega)$$
$$\bar{E}(k\omega) = -\hat{\bar{\alpha}}(k\omega)\hat{\bar{E}}(k\omega) \tag{A2}$$

Similarly, for n, $\hat{n}$, and $\check{n}$, we have in the magnetic free case

$$n(k\omega) = \frac{\hat{n}(k\omega)}{\varepsilon_L(k\omega)}$$
$$n(k\omega) = -\alpha_L(k\omega)\hat{n}(k\omega) \tag{A3}$$
$$= -\hat{\alpha}_L(k\omega)\hat{n}(k\omega)$$

where

$$\varepsilon_L(\bar{k}\omega) = \frac{\bar{k}\cdot\varepsilon(\bar{k}\omega)\cdot\bar{k}}{k^2}, \qquad \alpha_L(\bar{k}\omega) = \frac{\bar{k}\cdot\alpha(\bar{k}\omega)\cdot\bar{k}}{k^2} \tag{A4}$$

whereas n is related to $\bar{E}$, $\hat{\bar{E}}$, $\check{\bar{E}}$ as follows:

$$n(\bar{k}\omega) = -\frac{i\bar{k}}{4\pi e^2}\cdot\alpha(\bar{k}\omega)\bar{E}(\bar{k}\omega)$$
$$= -\frac{i\bar{k}}{4\pi e^2}\cdot\hat{\alpha}(k\omega)\hat{\bar{E}}(k\omega)$$
$$= \frac{ik}{4\pi e^2}\check{\bar{E}}(k\omega) \tag{A5}$$

Equations (A1)-(A5) are used to derive the coupling projection operator as follows. We first consider the unmagnetized case, but later generalize it to the magnetized case. Equation (2) can be expressed as

$$E_k^L = -\alpha_L^0(\bar{E}_k^L + \bar{E}_k^{\text{corr}}) \tag{A6}$$

where α_L^0 is the uncorrelated longitudinal polarizability tensor for the longitudinal component of the field. We have

$$\bar{E}_k^L + \bar{E}_k^{\text{corr}} = ik\psi_k n_k \tag{A7}$$

In the STLS theory, the correlational contribution is given by

$$\bar{E}_k^{\text{corr}} = ikn_k\phi_k u_k \tag{A8}$$

The superscript corr means "correlated," ψ_k is the static effective potential, ϕ_k is the longitudinal Coulomb potential, and μ_k is the effective screening function. The ψ_k, ϕ_k, and μ_k are interrelated as follows:

$$\psi_k = (1+u_k)\phi_k \tag{A9}$$

When equations (A6)-(A8) are combined and the result is compared with equation (A2) we obtain

$$\alpha_L = D^{-1}\alpha_L^0 \qquad \text{(A10)}$$

as "the strongly correlated longitudinal polarizability tensor," where D is given by

$$D = 1 + u_k\alpha_L^0 \qquad \text{(A11)}$$

For the case of magnetized plasmas, equation (A8) is still valid. When the longitudinal projection operator T is applied to the right-hand side of equation (A8), we obtain

$$i\bar{k}n_k\phi_k = \bar{k}\bar{k}\cdot \bar{E}/k^2 \qquad \text{(A12)}$$

where

$$T = \bar{k}\cdot \bar{k}/k^2 \qquad \text{(A13)}$$

Hence, equation (8) becomes

$$E_{k\nu}^{\text{corr}} = u_k k_\nu k_\mu / k^2 \qquad \text{(A14)}$$

When equation (A14) is substituted into equation (A7) and the result is compared with equation (A2) we find

$$\alpha_{\mu\nu} = D_{\mu\rho}^{-1}\alpha_{\rho\nu}^0 \qquad \text{(A15)}$$

as "the strongly coupled polarizability tensor," where

$$D_{\mu\nu} = \delta_{\mu\nu} + u_k\alpha_{\mu\rho}^0 k_\rho k_\mu / k^2 \qquad \text{(A16)}$$

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